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Computation of Parabolic Cylinder Functions by Means of a Tricomi Expansion

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Fast convergence expansion of the parabolic cylinder functions $U(a, x)$, $V(a, x)$, $W(a, \pm x)$ is obtained in terms of the Tricomi functions $E_\nu(z)$. The numerical results are quite accurate for a large interval of values of "a" and for $|x| \lesssim 7$. Tables are given for U and V in order to compare our results with other recent works on the same functions.

1. INTRODUCTION

Parabolic cylinder functions [1] find application in a number of physical problems, e.g., solution of the Schrödinger equation with parabolic potentials, which appear in fission barrier studies. Therefore, it is useful to develop methods for their computation with satisfactory accuracy over large regions of their arguments, in particular where standard power series expansions converge too slowly, and asymptotic formulae still give poor approximations.

This paper describes computations involving a fast convergent expansion of the Kummer confluent hypergeometric function due to Tricomi [2], which seems to be particularly suitable in the above-mentioned transition regions.

2. DESCRIPTION OF THE METHOD

The parabolic cylinder (or Weber) functions are solutions of the differential equation

$$\frac{d^2y}{dx^2} + (Ax^2 + Bx + C)\,y = 0, \tag{1}$$

where A, B, C are real numbers.

Equation (1) may be reduced to the two distinct standard forms

$$\frac{d^2y}{dx^2} - \left(\frac{1}{4}x^2 + a\right)y = 0, \tag{2}$$

$$\frac{d^2y}{dx^2} + \left(\frac{1}{4}x^2 - a\right)y = 0. \tag{3}$$

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Following the notations of Ref. [1], the general solution of (2) is a linear combination of the functions

$$\begin{aligned} U(a,x) = {} & \cos\left[\pi\left(\frac{1}{4}+\frac{a}{2}\right)\right]\frac{\Gamma\left(\frac{1}{4}-\frac{a}{2}\right)}{\pi^{1/2}\,2^{(a/2+1/4)}} \\ & \cdot\exp\left(\frac{-x^2}{4}\right)M\left(\frac{a}{2}+\frac{1}{4};\frac{1}{2};\frac{x^2}{2}\right) \\ & -\sin\left[\pi\left(\frac{1}{4}+\frac{a}{2}\right)\right]\frac{\Gamma\left(\frac{3}{4}-\frac{a}{2}\right)}{\pi^{1/2}\,2^{(a/2-1/4)}}\,x \\ & \cdot\exp\left(\frac{-x^2}{4}\right)M\left(\frac{a}{2}+\frac{3}{4};\frac{3}{2};\frac{x^2}{2}\right), \end{aligned} \tag{4a}$$

$$\begin{aligned} V(a,x) = {} & \frac{1}{\Gamma(\frac{1}{2}-a)}\left\{\sin\left[\pi\left(\frac{1}{4}+\frac{a}{2}\right)\right]\frac{\Gamma\left(\frac{1}{4}-\frac{a}{2}\right)}{\pi^{1/2}\,2^{(a/2+1/4)}}\right. \\ & \cdot\exp\left(\frac{-x^2}{4}\right)M\left(\frac{a}{2}+\frac{1}{4};\frac{1}{2};\frac{x^2}{2}\right) \\ & +\cos\left[\pi\left(\frac{1}{4}+\frac{a}{2}\right)\right]\frac{\Gamma\left(\frac{3}{4}-\frac{a}{2}\right)}{\pi^{1/2}\,2^{(a/2-1/4)}}\,x \\ & \left.\cdot\exp\left(\frac{-x^2}{4}\right)M\left(\frac{a}{2}+\frac{3}{4};\frac{3}{2};\frac{x^2}{2}\right)\right\}. \end{aligned} \tag{4b}$$

The solutions of (3) are linear combinations of $W(a,x)$ and $W(a,-x)$, defined as follows:

$$\begin{aligned} W(a,\pm x) = 2^{-3/4} & \left\{\left(\frac{G_1}{G_3}\right)^{1/2}\exp\left(-i\frac{x^2}{4}\right)M\left(\frac{1}{4}-i\frac{a}{2};\frac{1}{2};\frac{ix^2}{2}\right)\right. \\ & \left.\pm\left(\frac{2G_3}{G_1}\right)^{1/2}\cdot x\cdot\exp\left(-i\frac{x^2}{4}\right)M\left(\frac{3}{4}-i\frac{a}{2};\frac{3}{2};\frac{ix^2}{2}\right)\right\}. \end{aligned} \tag{5}$$

Here, $G_1 = |\Gamma(1/4+i(a/2))|$ and $G_3 = |\Gamma(3/4+i(a/2))| \cdot \Gamma(z)$ is the Euler gamma function, and $M(A;C;t)$ the Kummer confluent hypergeometric function.

The hypergeometric series which defines M is of practical use only over a restricted range of values A, C, t. Therefore, we preferred the fast convergence expansion derived by Tricomi [2]:

$$M(A;C;t) = \Gamma(C)\exp\left(\frac{t}{2}\right)\sum_{0}^{\infty}{}_n\, a_n\left(\frac{t}{2}\right)^n E_{C+n-1}(kt) \tag{6}$$

with C different from 0 or from a negative integer (as in our case, where $C=\frac{1}{2}$ or $\frac{3}{2}$), and $k=C/2-A$.

The Tricomi function $E_\nu(z)$, defined as follows:

$$E_\nu(z)=\sum_0^\infty{}_m \frac{(-1)^m}{\Gamma(\nu+m+1)}\frac{z^m}{m!} \tag{7}$$

is entire for any value of ν. The coefficients α_n are obtained through the recurrence relation

$$\alpha_{n+1}=\frac{1}{n+1}\left[(n+C-1)\,\alpha_{n-1}-2k\alpha_{n-2}\right] \tag{8}$$

for $n\geqslant 2$, while the first three α's are: $\alpha_0=1$, $\alpha_1=0$, $\alpha_2=C/2$.

It should be pointed out that $W(a, \pm x)$ is real for real arguments a, x, in spite of complex factors in (5). This is easily verified through the expansion (6):

$$\begin{aligned}\exp&\left(-i\frac{x^2}{4}\right)M\left(\frac{1}{4}-i\frac{a}{2};\frac{1}{2};i\frac{x^2}{2}\right)\\ &=\Gamma\left(\frac{1}{2}\right)\sum_0^\infty{}_n\,\alpha_n i^n\left(\frac{x^2}{4}\right)^n E_{n-1/2}\left(-\frac{ax^2}{4}\right),\end{aligned} \tag{9a}$$

$$\begin{aligned}\exp&\left(-i\frac{x^2}{4}\right)M\left(\frac{3}{4}-i\frac{a}{2};\frac{3}{2};i\frac{x^2}{2}\right)\\ &=\Gamma\left(\frac{3}{2}\right)\sum_0^\infty{}_n\,\tilde{\alpha}_n i^n\left(\frac{x^2}{4}\right)^n E_{n+1/2}\left(-\frac{ax^2}{4}\right).\end{aligned} \tag{9b}$$

It is now a simple matter to show by means of (8) that α_n and $\tilde{\alpha}_n$ are real when n is even and imaginary when it is odd.

As for $U(a,x)$ and $V(a,x)$, formulae (4a) and (4b) cannot be directly used when $a=n+\frac{1}{2}$ (n positive integer), because of the gamma functions. However, the trigonometric functions by which the Γ's are multiplied, eliminate any singularity. In fact, using the reflection and duplication formulae for Γ functions, (4a) and (4b) may be rewritten as

$$\begin{aligned}U(a,x)=&\frac{\pi^{1/2}\exp(-x^2/4)}{2^{(a/2+1/4)}\Gamma\left(\frac{3}{2}+\frac{a}{2}\right)}M\left(\frac{a}{2}+\frac{1}{4};\frac{1}{2};\frac{x^2}{2}\right)\\ &-\frac{\pi^{1/2}\exp(-x^2/4)\,x}{2^{(a/2-1/4)}\Gamma\left(\frac{1}{4}+\frac{a}{2}\right)}M\left(\frac{a}{2}+\frac{3}{4};\frac{3}{2};\frac{x^2}{2}\right),\end{aligned} \tag{10a}$$

$$V(a,x)=\frac{2^{(a/2+1/4)}}{\Gamma\left(\frac{3}{4}-\frac{a}{2}\right)}\sin\left[\pi\left(\frac{1}{4}+\frac{a}{2}\right)\right]\exp\left(-\frac{x^2}{4}\right)M\left(\frac{a}{2}+\frac{1}{4};\frac{1}{2};\frac{x^2}{2}\right)$$
$$+\frac{2^{(a/2+3/4)}}{\Gamma\left(\frac{1}{4}-\frac{a}{2}\right)}\cos\left[\pi\left(\frac{1}{4}+\frac{a}{2}\right)\right]\exp\left(-\frac{x^2}{4}\right)xM\left(\frac{a}{2}+\frac{3}{4};\frac{3}{2};\frac{x^2}{2}\right). \qquad (10b)$$

Evaluating a probability current density for a Schrödinger equation reduced to form (2) or (3) requires, in addition, the first derivatives $\partial U/\partial x$, $\partial V/\partial x$ or $(\partial W(a, \pm x)/\partial x)$, respectively. $\partial U/\partial x$ and $\partial V/\partial x$ are obtained through the recurrence relations (19.6) of Ref. [1]. $(\partial W(a, \pm x)/\partial x)$ cannot be obtained in such a simple way. Starting from the second derivative $\partial^2 W/\partial x^2$, expressed in terms of W through Eq. (3), and remembering that the Wronskian $\mathscr{W}\{W(a,x), W(a,-x)\}$ is equal to 1, $(\partial W(a,x)/\partial x)$ and $(\partial W(a,-x)/\partial x)$ for $x \neq 0$ were obtained as the solution of the system:

$$\begin{cases} \dfrac{\partial W}{\partial x}(a,x)+\dfrac{\partial W}{\partial x}(a,-x) \\ \quad =\displaystyle\int_{-x}^{+x}\frac{\partial^2 W}{\partial x^2}(a,x')\,dx'=\int_{-x}^{+x}\left(a-\frac{x'^2}{4}\right)W(a,x')\,dx' \\ -W(a,-x)\dfrac{\partial W}{\partial x}(a,x)+W(a,x)\dfrac{\partial W}{\partial x}(a,-x)=1. \end{cases} \qquad (11)$$

The sum at the left-hand side of the first Eq. (11) is due to the fact that $(\partial W(a,-x)/\partial x)$ is the first derivative of the mirror image of $W(a,x)$, i.e., $W(a,-x)$, calculated at point x. The integral of the right-hand side of the same equation was evaluated by changing the integration variable to $x''=x'/x$ and performing an 80-point Gauss–Legendre integration. Finally, when $x=0$

$$\frac{\partial W}{\partial x}(a,0)=-\frac{1}{2^{1/4}}\left(\frac{G_3}{G_1}\right)^{1/2} \qquad (12)$$

G_1 and G_3 have the same meaning as in Eq. (5).

3. Results and Comments

The values of U, V and W tabulated in [1] are reproduced by the expansion described in Section 2 without resorting to asymptotic formulae: "a" ranges from -5 to $+5$ and "x" from 0 to 5. Out of this region, a simple test of the method is possible

TABLE I

$U(-n-\frac{1}{2}, x)$ Calculated by Means of Hermite Polynomials and the Tricomi Expansion

a	$x=1$		$x=3$		$x=5$	
	Hermite	Tricomi	Hermite	Tricomi	Hermite	Tricomi
−0.5	0.778801	0.778801	0.105399	0.105399	0.193045 (−2)	0.193045 (−2)
−2.5	0.	0.1 (−19)	0.843195	0.843194	0.463309 (−1)	0.463309 (−1)
−4.5	−0.155760 (1)	−0.155760 (1)	0.316198 (1)	0.316198 (1)	0.922759	0.922757
−6.5	0.124608 (2)	0.124608 (2)	−0.101184 (2)	−0.101183 (2)	0.142082 (2)	0.142081 (2)
−8.5	−0.102802 (3)	−0.102802 (3)	−0.543862 (2)	−0.543860 (2)	0.142816 (3)	0.142815 (3)
−10.5	0.947026 (3)	0.947022 (3)	0.100172 (4)	0.100171 (4)	0.346866 (3)	0.346864 (3)
−12.5	−0.968834 (4)	−0.968828 (4)	−0.712587 (4)	−0.712583 (4)	−0.114660 (5)	−0.114659 (5)
−14.5	0.107513 (6)	0.107512 (6)	−0.182131 (5)	−0.182130 (5)	−0.457864 (5)	−0.457860 (5)
−16.5	−0.124708 (7)	−0.124707 (7)	0.166117 (7)	0.166116 (7)	0.226995 (7)	0.226994 (7)
−18.5	0.141034 (8)	0.141032 (8)	−0.354970 (8)	−0.354967 (8)	−0.717090 (7)	−0.717084 (7)
−20.5	−0.126116 (9)	−0.126115 (9)	0.485598 (9)	0.485594 (9)	−0.608556 (9)	−0.608550 (9)
−22.5	−0.314645 (9)	−0.314642 (9)	−0.205025 (10)	−0.205023 (10)	0.124619 (11)	0.124617 (11)
−24.5	0.721101 (11)	0.721093 (11)	−0.150538 (12)	−0.150536 (12)	0.319162 (11)	0.319160 (11)
−26.5	−0.328760 (13)	−0.328756 (13)	0.715325 (13)	0.715316 (13)	−0.764495 (13)	−0.764485 (13)
−28.5	0.124084 (15)	0.124082 (15)	−0.216894 (15)	−0.216891 (15)	0.193313 (15)	0.193312 (15)
−30.5	−0.446327 (16)	−0.446321 (16)	0.500304 (16)	0.500297 (16)	−0.406437 (15)	−0.406464 (15)

TABLE II

Parabolic Cylinder Functions and Their Derivatives for $a = 0.6$

x	$U(a,x)$ [3]	$U(a,x)$ This work	$U'(a,x)$ [3]	$U'(a,x)$ This work	$V(a,x)$ This work	$V'(a,x)$ This work
0.2	0.1041035 (1)	0.1045838 (1)	−0.9815019	−0.9215545	0.7467478	0.1049070
0.4	0.8706166	0.8737212	−0.7805672	−0.8022356	0.7770704	0.1997101
0.6	0.7235714	0.7240349	−0.6905327	−0.6966092	0.8273913	0.3059468
0.8	0.5944484	0.5943708	−0.6009693	−0.6016246	0.9006809	0.4037291
1.0	0.4828850	0.4827970	−0.5157178	−0.5154899	0.1001527 (1)	0.5832831
1.2	0.3876907	0.3876484	−0.4374586	−0.4372695	0.1136668 (1)	0.7761013
1.4	0.3074046	0.3073891	−0.3666450	−0.3665590	0.1315797 (1)	0.1026605 (1)
1.6	0.2405373	0.2405328	−0.3032610	−0.3032300	0.1552786 (1)	0.1359620 (1)
1.8	0.1856086	0.1856077	−0.2472490	−0.2472398	0.1867509 (1)	0.1811142 (1)
2.0	0.1411531	0.1411532	−0.1985035	−0.1985016	0.2288620 (1)	0.2434163 (1)
2.2	0.1057368	0.1057371	−0.1568084	−0.1568085	0.2857794 (1)	0.3307809 (1)
2.4	0.7798338 (−1)	0.7798356 (−1)	−0.1218032	−0.1218036	0.3636298 (1)	0.4551863 (1)
2.6	0.5660288 (−1)	0.5660298 (−1)	−0.9298441 (−1)	−0.9298471 (−1)	0.4715267 (1)	0.6350140 (1)
2.8	0.4041827 (−1)	0.4041832 (−1)	−0.6973348 (−1)	−0.6973366 (−1)	0.6232010 (1)	0.8988589 (1)
3.0	0.2838443 (−1)	0.2838445 (−1)	−0.5135743 (−1)	−0.5135753 (−1)	0.8396199 (1)	0.1291822 (2)
3.2	0.1959847 (−1)	0.1959848 (−1)	−0.3713419 (−1)	−0.3713424 (−1)	0.1153250 (2)	0.1886034 (2)
3.4	0.1330129 (−1)	0.1330130 (−1)	−0.2635438 (−1)	−0.2635441 (−1)	0.1615088 (2)	0.2798505 (2)
3.6	0.8871506 (−2)	0.8871510 (−2)	−0.1835510 (−1)	−0.1835511 (−1)	0.2306430 (2)	0.4221794 (2)
3.8	0.5813600 (−2)	0.5813602 (−2)	−0.1254343 (−1)	−0.1254344 (−1)	0.3358819 (2)	0.6477450 (2)
4.0	0.3742474 (−2)	0.3742475 (−2)	−0.8409576 (−2)	−0.8409579 (−2)	0.4988395 (2)	0.1011046 (3)
4.2	0.2366291 (−2)	0.2366291 (−2)	−0.5530713 (−2)	−0.5530715 (−2)	0.7555850 (2)	0.1605855 (3)
4.4	0.1469296 (−2)	0.1469297 (−2)	−0.3567767 (−2)	−0.3567768 (−2)	0.1167268 (3)	0.2596006 (3)
4.6	0.8958306 (−3)	0.8958307 (−3)	−0.2257280 (−2)	−0.2257280 (−2)	0.1839230 (3)	0.4272221 (3)
4.8	0.5362478 (−3)	0.5362479 (−3)	−0.1400613 (−2)	−0.1400614 (−2)	0.2955909 (3)	0.7158554 (3)
5.0	0.3151240 (−3)	0.3151240 (−3)	−0.8522549 (−3)	−0.8522550 (−3)	0.4845569 (3)	0.1221482 (4)
6.0	0.1668540 (−4)	0.1668538 (−4)	−0.5295820 (−4)	−0.5295829 (−4)	0.7723948 (4)	0.2330409 (5)
7.0	0.5502266 (−6)	0.5515030 (−6)	−0.2008894 (−5)	−0.2003818 (−5)	0.2023700 (6)	0.7112426 (6)

TABLE III

Approximate and Exact Calculations of $U(-0.5, x)$ and $V(+0.5, x)$

x	$U(-0.5, x)$			$V(+0.5, x)$		
	Exact	[3]	This work	Exact	[4]	This work
2.5	0.209611	0.209611	0.209611	3.80649	3.80622	3.80649
2.6	0.184519	0.184519	0.184519	4.32412	4.32388	4.32412
2.7	0.161621	0.161621	0.161621	4.93676	4.93655	4.93676
2.8	0.140858	0.140858	0.140858	5.66444	5.66426	5.66444
2.9	0.122151	0.122151	0.122151	6.53197	6.53181	6.53197
3.0	0.105399	0.105399	0.105399	7.57012	7.56998	7.57012
3.1	0.904914 (−1)	0.904913 (−1)	0.904914 (−1)	8.81724	8.81712	8.81724
3.2	0.773047 (−1)	0.773047 (−1)	0.773047 (−1)	1.03213 (1)	1.03212 (1)	1.03213 (1)
3.3	0.657103 (−1)	0.657102 (−1)	0.657103 (−1)	1.21425 (1)	1.21424 (1)	1.21425 (1)
3.4	0.555762 (−1)	0.555762 (−1)	0.555762 (−1)	1.43566 (1)	1.43565 (1)	1.43566 (1)
3.5	0.467706 (−1)	0.467706 (−1)	0.467706 (−1)	1.70595 (1)	1.70594 (1)	1.70595 (1)
3.6	0.391639 (−1)	0.391639 (−1)	0.391639 (−1)	2.03730 (1)	2.03729 (1)	2.03730 (1)
3.7	0.326308 (−1)	0.326307 (−1)	0.326308 (−1)	2.44519 (1)	2.44518 (1)	2.44519 (1)
3.8	0.270518 (−1)	0.270518 (−1)	0.270518 (−1)	2.94946 (1)	2.94946 (1)	2.94946 (1)
3.9	0.223149 (−1)	0.223149 (−1)	0.223149 (−1)	3.57557 (1)	3.57556 (1)	3.57557 (1)
4.0	0.183156 (−1)	0.183156 (−1)	0.183156 (−1)	4.35630 (1)	4.35629 (1)	4.35630 (1)

half-integer values of "a" in the cases of U and V, which are then expressible in terms of Hermite polynomials:

$$U(-n-\tfrac{1}{2}, x) = 2^{-n/2} \exp(x^2/4)\, H_n(x/2^{1/2}), \tag{13a}$$

$$V(n+\tfrac{1}{2}, x) = 2^{-n/2} \exp(x^2/4)(-i)^n\, H_n(ix/2^{1/2}). \tag{13b}$$

As an example, Table I contains values of $U(-n-\frac{1}{2}, x)$ computed through formulae (10a) and (13a) for a few values of x ($\leqslant 5$): the agreement is excellent even for n as high as 30. For high positive values of x, some years ago Latham and Redding [3] solved an integral representation of $U(a, x)$ in the range $-0.5 < a \leqslant 0.8$ by means of an algorithm previously applied to Airy functions. In their Tables I and II x ranged from 0.2 to 16, but this approximation turned out to be very accurate for $x > 1 \div 2$ only. Our Table II shows a comparison of results obtained by the procedure [3] and by Tricomi's expansion for $a = 0.6$ and $x \leqslant 7$. In fact, our method fails for higher values of x, but one can here resort to asymptotic expansions: those given by Miller in Section (19.8) of Ref. [1] are already accurate in this case.

In a subsequent note [4], Latham and Redding studied $V(a, x)$, relating it to $U(a, x)$ and $U(a, -x)$ by means of a formula given in [1]. Table III shows a comparison of results obtained in [3, 4] and in this work for the trivial cases $U(-0.5, x)$ and $V(0.5, x)$.

Summing up, Tricomi's expansion allows accurate evaluation of the parabolic cylinder functions $U(a, x)$, $V(a, x)$, $W(a, \pm x)$ and of their derivatives for a large interval of values of "a" and for $|x| \lesssim 7$. Higher $|x|$ values require asymptotic expansions.

APPENDIX A. A Relation between Tricomi and Airy Functions in Asymptotic Formulae of U, V, W

Asymptotic formulae of the parabolic cylinder functions for large $|a|$ values are given in [1] by means of the Airy functions $Ai(t)$ and $Bi(t)$.

Even if in these cases the expansion described in Section 2 works better than the asymptotic formulae for $|x| \lesssim 7$, it may be of some interest to express them by means of the Tricomi functions (7), which allow simple and fast computation of $Ai(t)$ and $Bi(t)$.

Let us define the following auxiliary variables:

$$\xi = x/(2\,|a|^{1/2}); \qquad t = (4\,|a|)^{2/3}\,\tau;$$

$$\tau = -(\tfrac{3}{2}\theta)^{2/3}, \quad \text{with } \theta = \tfrac{1}{4}[\arccos \xi - \xi(1-\xi^2)^{1/2}] \quad \text{for } \xi < 1$$

$$\tau = +(\tfrac{3}{2}\theta)^{2/3}, \quad \text{with } \theta = \tfrac{1}{4}\{\xi(\xi^2-1)^{1/2} - \ln[\xi + (\xi^2-1)^{1/2}]\} \quad \text{for } \xi > 1.$$

Then, for "a" large and negative, and $0 \leqslant x < \infty$:

$$U(a,x) \simeq \frac{1}{2^{(1/4+a/2)}} \Gamma\left(\frac{1}{4}+\frac{|a|}{2}\right)\left(\frac{t}{\xi^2-1}\right)^{1/4} Ai(t), \tag{14a}$$

$$V(a,x) \simeq \frac{1}{2^{(1/4+a/2)}} \frac{\Gamma\left(\frac{1}{4}+\frac{|a|}{2}\right)}{\Gamma(\frac{1}{2}+|a|)}\left(\frac{t}{\xi^2-1}\right)^{1/4} Bi(t), \tag{14b}$$

while, for "a" large and positive and $0 \leqslant x < \infty$:

$$W(a,x) \simeq \left(\frac{\pi}{2}\right)^{1/2} a^{-1/4} \exp\left(-\frac{\pi a}{2}\right)\left(\frac{t}{\xi^2-1}\right)^{1/4} Bi(-t), \tag{15a}$$

$$W(a,-x) \simeq 2\left(\frac{\pi}{2}\right)^{1/2} a^{-1/4} \exp\left(\frac{\pi a}{2}\right)\left(\frac{t}{\xi^2-1}\right)^{1/4} Ai(-t). \tag{15b}$$

In these cases $\partial W(a, \pm x)/\partial x$ may be obtained directly from derivation of (15a), (15b) instead of using the numerical procedure described in Section 2:

$$\frac{\partial W}{\partial x}(a,x) = \left(\frac{1}{4t}\frac{dt}{dx} - \frac{1}{4a^{1/2}}\frac{\xi}{\xi^2-1}\right) W(a,x) - \left(\frac{\pi}{2}\right)^{1/2} a^{-1/4} \exp\left(-\frac{\pi a}{2}\right)\left(\frac{t}{\xi^2-1}\right)^{1/4} \frac{dt}{dx} Bi'(-t), \tag{16a}$$

$$\frac{\partial W}{\partial x}(a,-x) = \left(\frac{1}{4t}\frac{dt}{dx} - \frac{1}{4a^{1/2}}\frac{\xi}{\xi^2-1}\right) W(a,-x) - 2\left(\frac{\pi}{2}\right)^{1/2} a^{-1/4} \exp\left(\frac{\pi a}{2}\right)\left(\frac{t}{\xi^2-1}\right)^{1/4} \frac{dt}{dx} Ai'(-t). \tag{16b}$$

Here, $Bi'(-t) = d\,Bi(-t)/d(-t) = -d\,Bi(-t)/dt$, and similarly for $Ai'(-t)$.

The Airy functions and their derivatives may be rewritten in terms of the Tricomi functions (7). Let us define $\chi = |t|^{3/2}/3$. Then, for $t < 0$:

$$Ai(t) = \frac{(|t|)^{1/2}}{3}\left[\chi^{-1/3} E_{-1/3}(\chi^2) + \chi^{1/3} E_{1/3}(\chi^2)\right], \tag{17a}$$

$$Bi(t) = \left(\frac{|t|}{3}\right)^{1/2}\left[\chi^{-1/3} E_{-1/3}(\chi^2) - \chi^{1/3} E_{1/3}(\chi^2)\right]. \tag{17b}$$

For $t > 0$:

$$Ai(t) = \frac{(|t|)^{1/2}}{3}\left[\chi^{-1/3} E_{-1/3}(-\chi^2) - \chi^{1/3} E_{1/3}(-\chi^2)\right], \tag{18a}$$

$$Bi(t) = \left(\frac{|t|}{3}\right)^{1/2}\left[\chi^{-1/3} E_{-1/3}(-\chi^2) + \chi^{1/3} E_{1/3}(-\chi^2)\right]. \tag{18b}$$

Conversely, $Ai(-t)$ and $Bi(-t)$ are given by formulae (17a), (17b) for $t > 0$ and by formulae (18a), (18b) for $t < 0$. As for their derivatives, to be used in (16a) and (16b), when $t < 0$

$$Ai'(-t) = \frac{t}{3}\,[\chi^{-2/3}E_{-2/3}(-\chi^2) - \chi^{2/3}E_{2/3}(-\chi^2)], \tag{19a}$$

$$Bi'(-t) = \frac{-t}{3^{1/2}}\,[\chi^{-2/3}E_{-2/3}(-\chi^2) + \chi^{2/3}E_{2/3}(-\chi^2)]. \tag{19b}$$

For $t > 0$:

$$Ai'(-t) = \frac{-t}{3}\,[\chi^{-2/3}E_{-2/3}(\chi^2) - \chi^{2/3}E_{2/3}(\chi^2)] \tag{20a}$$

$$Bi'(-t) = \frac{t}{3^{1/2}}\,[\chi^{-2/3}E_{-2/3}(\chi^2) + \chi^{2/3}E_{2/3}(\chi^2)]. \tag{20b}$$

In the case $\xi = 1$, $t = 0$, formulae (14a) to (16b) cannot be utilized. Keeping in mind the following limits:

$$\lim_{\xi\to 1}\left(\frac{t}{\xi^2 - 1}\right)^{1/4} = |a|^{1/6};$$

$$\lim_{\xi\to 1}\frac{1}{4}\left(\frac{1}{t}\frac{dt}{dx} - \frac{1}{a^{1/2}}\frac{\xi}{\xi^2 - 1}\right) = -\frac{1}{20a^{1/2}},$$

one gets

$$U(a, 2|a|^{1/2}) \simeq 2^{-1/4-a/2}\Gamma\left(\frac{1}{4} + \frac{|a|}{2}\right)|a|^{1/6}Ai(0), \tag{21a}$$

$$V(a, 2|a|^{1/2}) \simeq 2^{-1/4-a/2}\frac{\Gamma\left(\frac{1}{4} + \frac{|a|}{2}\right)}{\Gamma(\frac{1}{2} + |a|)}|a|^{1/6}Bi(0), \tag{21b}$$

$$W(a, 2a^{1/2}) \simeq \left(\frac{\pi}{2}\right)^{1/2} a^{-1/4}\exp\left(-\frac{\pi a}{2}\right)a^{1/6}Bi(0), \tag{22a}$$

$$W(a, -2a^{1/2}) \simeq 2\left(\frac{\pi}{2}\right)^{1/2} a^{-1/4}\exp\left(\frac{\pi a}{2}\right)a^{1/6}Ai(0), \tag{22b}$$

$$\frac{\partial W}{\partial x}(a, 2a^{1/2}) = \frac{-1}{20a^{1/2}}W(a, 2a^{1/2}) - \left(\frac{\pi}{2}\right)^{1/2} a^{-1/4}\exp\left(\frac{-\pi a}{2}\right)a^{1/3}Bi'(0), \tag{23a}$$

$$\frac{\partial W}{\partial x}(a, -2a^{1/2}) = \frac{-1}{20a^{1/2}}W(a, -2a^{1/2}) - 2\left(\frac{\pi}{2}\right)^{1/2} a^{-1/4}\exp\left(\frac{\pi a}{2}\right)a^{1/3}Ai'(0). \tag{23b}$$

Remember that $a < 0$ in (21a) and (21b) and $a > 0$ in (22a) to (23b). The values of the Airy functions and of their derivatives at $t = 0$ are

$$Ai(0) = (\tfrac{1}{3})^{2/3}\,\Gamma(\tfrac{2}{3}); \qquad Bi(0) = 3^{1/2} Ai(0);$$

$$Ai'(0) = \frac{1}{3^{1/3}\Gamma(\tfrac{1}{3})}; \qquad Bi'(0) = -(3)^{1/2}\,Ai'(0).$$

Note. The FORTRAN program TRISE, to compute parabolic cylinder functions and their derivatives, is available on request.

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